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Quantum state reconstruction by entangled measurements

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Abstract. We address quantum state reconstruction for d-dimensional systems based on measuring, on the system of interest and a probe, of a single entangled observable defined on the bipartite system/probe Hilbert space. We show that the statistics of the measurement and the knowledge of the probe preparation suffice to reliably reconstruct the density matrix of the system, as well as the expectation value of any desired operator, including those not corresponding to observable quantities. The statistical robustness of the reconstruction is examined and a method is developed to minimize statistical errors by tuning the probe preparation. Numerical simulations of the whole reconstruction procedure are also presented for qubit systems.

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QICS. 03.10.+m Entanglement measures – 25.40.+t Automated state and process tomography

1 Introduction

The *state* of a physical system is a mathematical tool providing the complete information on the system itself, i.e. the capability to predict the result of any possible measurement performed on the system. In classical mechanics it is always possible, at least theoretically, to devise a proper set of measurements that fully recovers the state of a system. In quantum mechanics, on the other hand, the problem is more complex owing to limitations posed by Heisenberg uncertainty relations [1,2] and by the nocloning theorem [3–6]. Indeed, it is neither possible to perform a set of measurements on the system without altering its initial state, nor is possible to create a copy of the system without knowing in advance its state. As a consequence, there is no way to recover the quantum state of a single system without any prior knowledge of it [7].

A solution to this problem is *quantum tomography*, whose basic principles have been earlier introduced by Fano [8]. Tomography is based on repeated multiple measurements performed on different preparations (copies) of the the unknown state. These measurements are extracted from an appropriate set of observables called *quorum of observables* [8,9]. Quantum tomography has been extensively developed in the last decade [10,11] and experimentally implemented in several different physical systems [12].

As a matter of fact, implementation of quantum tomography needs the measurements of a quorum, i.e. a complete set of observables. In turn, for system with many degrees of freedom, it may be not obvious how to measure all of the necessary observables. For this reason, in this paper we address a different method that, while still in need of multiple copies of the unknown state, nevertheless permits quantum state reconstruction by the measurement of a *single observable*. This result is achieved by jointly measuring the system of interest (the *signal*) and a second system (the *probe*) whose state is known and under control of the observer. The measured observable is defined on the bipartite Hilbert space describing the signal/probe joint system. Upon choosing an *entangled measure*, i.e. an observable that cannot be factorized in two separate measurements on the system and the probe respectively, it is possible to reconstruct the density matrix of the system, as well as the expectation value of any desired operator, starting from the statistics of the measurement and the knowledge of the probe preparation (see Fig. 1).

The idea of using entangled measure in a system/probe configuration has its root in the field of indirect measurements for infinite dimensional systems [13,14] with applications to the joint measurements of non commuting observables [15–17] and to operational phase measurements [18–20]. More recently, it has been applied to finite dimensional systems in order to analyze the information/disturbance trade-off [21,22], to realize measurements by programmable quantum processors [23], and to build a *universal* detector [24]. In this paper, we further develop the idea of measuring a single entangled observable instead of a quorom [24] and focus our attention on the explicit formulas connecting the statistics of the

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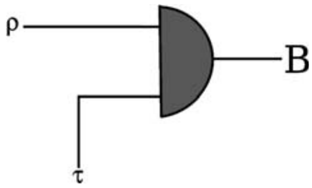

Fig. 1. Schematic diagram of state reconstruction by entangled measurements. The system of interest (the signal ρ) and a second system (the *probe* τ) whose state is known and under control of the observer are jointly measured. The observable is defined on the bipartite Hilbert space describing the signal/probe joint system. Upon choosing an *entangled measure* B, i.e. an observable that cannot be factorized in two separate measurements on the system and the probe respectively, it is possible to reconstruct the quantum state of the system.

measurement to the system matrix elements, as well as other expectation values, and on the minimization of statistical errors by suitably tuning the probe preparation. Indeed reducing errors due to statistical fluctuations is a relevant issue for experimental implementations. Very recently the realization of a quasi complete Bell measurement (3 over four results) has been reported [25] which, besides information protocols, will find application in the field of state reconstruction.

The paper is structured as follows: in the next section we establish notation and introduce some background material which will be used throughout the paper. In Section 3 we deal with state reconstruction by entangled measurements on qubit systems, whereas in Section 4 the procedure is generalized to arbitrary finite dimension. In Section 5 we extend the method to the reconstruction of the expectation value of a generic operator, including those not corresponding to observable quantities. In Section 6 the statistical errors in the reconstruction are evaluated and their minimization by tuning of the probe preparation is addressed. In addition, we report the results of Monte Carlo simulated experiments illustrating the application of the whole procedure. Finally, Section 7 closes the paper with some concluding remarks.

2 The Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$

We denote by $\mathcal{H}_1$ the Hilbert space, of dimension d_1, of the signal system. Analogously, $\mathcal{H}_2$, of dimension d_2, is the Hilbert space of the probe. For our purposes it will suffice to consider $d_1 = d_2 = d$, though the notation and the results of the present section are valid also for $d_1 \neq d_2$. We write state-vectors in $\mathcal{H}_1 \otimes \mathcal{H}_2$ by the double-ket $|\psi\rangle\rangle$, in order to distinguish states of the bipartite system from single system ones. A generic bipartite state $|\psi\rangle\rangle$ can be written as

$$|\psi\rangle\rangle = \sum_{i=0}^{d_1-1} \sum_{j=0}^{d_2-1} \psi_{ij} |i\rangle_1 \otimes |j\rangle_2 \equiv |\Psi\rangle\rangle \tag{1}$$

where $|i\rangle_1$ and $|j\rangle_2$ are orthonormal bases for $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Once these bases are fixed the state $|\psi\rangle\rangle$ is unequivocally determined by the matrix $\Psi = [[\psi_{ij}]]$ [26,27]. In turn, this notation is sometimes referred to as *matrix notation*. Analogously, any linear operator $A \in \mathcal{L}[\mathcal{H}_1 \to \mathcal{H}_2]$ can be written as

$$A = \sum_{i=0}^{d_1-1} \sum_{j=0}^{d_2-1} a_{ij} |j\rangle_2 \, {}_1\langle i|. \tag{2}$$

Once bases in $\mathcal{H}_1$ and $\mathcal{H}_2$ are fixed, one can easily show the isomorphism between the two *linear spaces* $\mathcal{H}_1 \otimes \mathcal{H}_2$ and $\mathcal{L}[\mathcal{H}_1 \to \mathcal{H}_2]$. In addition, $\forall\, |\psi\rangle\rangle, |\phi\rangle\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ one has

$$\langle\langle\psi|\phi\rangle\rangle = \mathrm{Tr}\left[\Psi^\dagger \Phi\right], \tag{3}$$

i.e. the isomorphism is established between the *Hilbert spaces* $\mathcal{H}_1 \otimes \mathcal{H}_2$ and $\mathcal{L}[\mathcal{H}_1 \to \mathcal{H}_2]$. A relevant consequence of the isomorphism is the following formula, which holds for any vector $|\psi\rangle\rangle$ in $\mathcal{H}_1 \otimes \mathcal{H}_2$ and any factorized operator $A \otimes B$

$$\begin{aligned} A \otimes B|\psi\rangle\rangle &= \sum_{ij} \psi_{ij} \left(\sum_{lm} a_{lm} |l\rangle_{21}\langle m|i\rangle_1 \right) \\ &\quad \otimes \left(\sum_{hk} b_{hk} |h\rangle_{12}\langle k|j\rangle_2 \right) \\ &= \sum_{hl} \left(\sum_{ij} a_{li} \psi_{ij} b_{hj} \right) |h\rangle_1 \otimes |l\rangle_2 \\ &= \sum_{ij} \left(A \Psi B^T\right)_{ij} |i\rangle \otimes |j\rangle = |A\Psi B^T\rangle\rangle. \end{aligned} \tag{4}$$

Equation (4) will be used throughout the paper.

3 State reconstruction for qubits

In this section we describe state reconstruction by entangled measurements on qubits. As we will see a bidimensional probe suffice to obtain reconstruction, and therefore $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{C}^2$. A convenient description of signal state is obtained by its density operator in the Bloch sphere representation i.e.

$$\varrho = \frac{1}{2} \sum_{j=0}^{3} s_j \sigma_j \tag{5}$$

where $\sigma_0 = \mathbb{I}$ is the identity matrix, $\{\sigma_k\}$,$k = 1, 2, 3$ are the Pauli matrices, $s_0 = 1$ and the three real numbers s_1, s_2, s_3 are the coefficients fully specifying the state, which obey the relation $\sum_{j=1}^{3} s_j^2 \leq 1$. Analogously, the state of the probe is written as

$$\tau = \frac{1}{2} \sum_{j=0}^{3} t_j \sigma_j, \qquad t_0 = 1. \tag{6}$$

In order to describe entangled measure we introduce the so-called *Bell basis* [28] for the bipartite space $\mathcal{H}_1 \otimes \mathcal{H}_2$, which is composed by the maximally entangled states

$$|\psi_j\rangle\rangle = |\tfrac{1}{\sqrt{2}}\sigma_j\rangle\rangle \quad j = 0, 1, 2, 3. \tag{7}$$

The Bell basis is orthonormal and corresponds to the spectral measure of an observable, referred to as the *Bell observable* $B = \sum_{j=0}^{3} \lambda_j \, |\psi_j\rangle\rangle\langle\langle\psi_j|$, We now assume to have an experimental apparatus measuring the Bell observables [29], i.e. sampling the probabilities $\{p_j\}$ of the outcome λ_j and focus attention on obtaining a relation giving these probabilities in terms of the matrix elements of ϱ and τ.

The signal and the probe are initially uncorrelated, thus the system is described by the tensor product $\varrho \otimes \tau$. The probabilities $\{p_j\}$ are given by

$$\begin{aligned} p_j &= \mathrm{Tr}\left[\varrho \otimes \tau |\frac{1}{\sqrt{2}}\sigma_j\rangle\rangle\langle\langle\frac{1}{\sqrt{2}}\sigma_j|\right] = \langle\langle\frac{1}{\sqrt{2}}\sigma_j|\varrho \otimes \tau|\frac{1}{\sqrt{2}}\sigma_j\rangle\rangle \\ &= \langle\langle\frac{1}{\sqrt{2}}\sigma_j|\frac{1}{\sqrt{2}}\varrho\sigma_j\tau^T\rangle\rangle = \mathrm{Tr}\left[\frac{1}{2}\sigma_j^\dagger\varrho\sigma_j\tau^T\right] \\ &= \mathrm{Tr}\left[\frac{1}{2}\sigma_j\varrho\sigma_j\tau^T\right], \end{aligned} \tag{8}$$

and using the Bloch description

$$p_j = \frac{1}{8}\sum_{kl} s_k t_l \mathrm{Tr}\left[\sigma_j\sigma_k\sigma_j\sigma_l^T\right]. \tag{9}$$

Our aim is now to invert equation (9) to reconstruct the unknown coefficients s_k starting from the statistics of the measurement (the probabilities p_j) and from the state of the probe (the three coefficients t_l). At first we recall the relations $\sigma_0\sigma_j = \sigma_j\sigma_0 = \sigma_j$, $\sigma_j^2 = \sigma_j\sigma_j = \mathbb{I} = \sigma_0$, and $\sigma_i\sigma_j = \epsilon_{ijk} i\sigma_k$, $i \neq j$, ϵ_{ijk} being the elements of the fully antisymmetric tensor. Concerning the product of three Pauli matrices $\sigma_j\sigma_k\sigma_j$ we have the result

$$\sigma_j\sigma_k\sigma_j = (-1)^{1+\delta_{jk}+\delta_{0k}+\delta_{0j}}\,\sigma_k, \quad k = 0,1,2,3 \tag{10}$$

which follows follows from

$$\begin{aligned} \sigma_j\sigma_k\sigma_j &= \epsilon_{jkl} i\sigma_l\sigma_j = \epsilon_{jkl} i\epsilon_{ljm} i\sigma_m \\ &= \epsilon_{jkl}\epsilon_{ljm} i^2\sigma_m = -\sigma_k, \quad k = 1,2,3. \end{aligned} \tag{11}$$

Using equations (10) and the relation $\sigma_l^T = (-1)^{\delta_{2l}}\sigma_l$ we rewrite equation (9) as

$$\begin{aligned} p_j &= \frac{1}{8}\sum_{kl} s_k t_l \mathrm{Tr}\left[\sigma_j\sigma_k\sigma_j\sigma_l^T\right] = \frac{1}{8}\sum_{k} s_k t_k \mathrm{Tr}\left[\sigma_j\sigma_k\sigma_j\sigma_k^T\right] \\ &= \frac{1}{4}\sum_{k} s_k t_k \left(-\right)^{1+\delta_{jk}+\delta_{k0}+\delta_{j0}+\delta_{k2}}. \end{aligned} \tag{12}$$

Equation (12) is a linear system with unknowns $\{s_j\}$, which can be solved upon assuming the condition $t_k \neq 0 \; \forall k$ i.e. the probe state should be *distributed* in the whole Hilbert space with nonzero component in any subspace. Inverting equation (12) we arrive at

$$s_j = \frac{1}{t_j}(-1)^{1+\delta_{j0}+\delta_{j2}}\sum_{k=0}^{3}(-1)^{1+\delta_{k0}}p_k. \tag{13}$$

The case $j = 0$ in equation (13) is just the normalization of the probabilities, while the others provide the three coefficients $\{s_j\}$ in terms of the coefficients $\{t_k\}$, fixed by the observer, and of the experimental probabilities $\{p_k\}$. Notice that a similar derivation may be easily obtained for different Bell measurements, based on different projectors over maximally entangled states, whereas for factorized measurements there is no way to reliably recover the signal density matrix. Notice, however, that classically correlated measurements, i.e. factorized measurements assisted by classical communication between system and probe, may be devised for state reconstruction upon a suitable choice of the communication protocol [30].

4 State reconstruction for qudits

The method outlined in the previous Section can be generalized to systems described by a Hilbert space of arbitrary *finite* dimension d (qudit). To this aim we introduce a convenient d-dimensional representation for the signal and the probe states, which generalizes the Pauli matrix description and will be used to define the Bell observable as well.

Let us consider the set of d^2 unitary transformations $U(n,m)$: $\mathbb{C}^d \to \mathbb{C}^d$ defined by

$$U(n,m) = \sum_{k=0}^{d-1} e^{\frac{2\pi i}{d}kn}|k\rangle\langle k \oplus m| \tag{14}$$

where n and m range in $0, d-1$ and $\oplus$ denotes the addition modulo d. The unitaries $U(n,m)$ form a unitary irreducible non-Abelian representation of the Abelian group $\frac{\mathbb{Z}}{d\mathbb{Z}} \times \frac{\mathbb{Z}}{d\mathbb{Z}}$. In order to use these unitaries as a basis for $\mathcal{L}\left[\mathbb{C}^d \to \mathbb{C}^d\right]$ we have to check trace, composition law and orthogonality (see Appendix):

1. **trace:** $U(n,m)$ are zero trace matrices except for $n,m = 0$; $U(0,0) = \mathbb{I}$, $\mathrm{Tr}\left[U(0,0)\right] = d$;
2. **composition law:**

$$U(n,m)U(p,q) = e^{\frac{2\pi i}{d}mp}\,U(n \oplus p, m \oplus q); \tag{15}$$

3. **orthogonality:**

$$\begin{aligned} \langle\langle U(p,q)|U(n,m)\rangle\rangle &= \mathrm{Tr}\left[U^\dagger(p,q)U(n,m)\right] \\ &= d\delta_{np}\delta_{mq}. \end{aligned} \tag{16}$$

Analogously, one can easily show that the states $|U(n,m)/\sqrt{d}\rangle\rangle$ form an orthogonal, maximally entangled basis for the bipartite space $\mathbb{C}^d \otimes \mathbb{C}^d$. We can now define a Bell observable in dimension d as

$$B = \sum_{kl} \lambda_{kl} \varPi_{kl} \tag{17}$$

where

$$\varPi_{kl} = \frac{1}{d}|U(k,l)\rangle\rangle\langle\langle U(k,l)|. \tag{18}$$

The state of the system can be expressed in terms of the unitaries $U(n,m)$ by generalizing the Bloch description as follows

$$\varrho = \frac{1}{d}\sum_{n=0}^{d-1}\sum_{m=0}^{d-1} s_{nm}\, U(n,m). \tag{19}$$

We have to impose some conditions on the coefficients s_{nm} such that ϱ in equation (19) describe a density matrix. Unit trace imposes $s_{00} = 1$ whereas hermiticity, using equation (51), rewrites as

$$\sum_{nm} s_{nm} U(n,m) = \sum_{pq} s^*_{pq} U^\dagger(p,q) = \sum_{pq} s^*_{pq} e^{\frac{2\pi i}{d}pq}\, U(-p,-q)$$

leading to

$$s_{nm} = s^*_{-n,-m} e^{\frac{2\pi i}{d}nm} = s^*_{d-n,d-m} e^{\frac{2\pi i}{d}nm}. \tag{20}$$

In addition, since for any density operator $\text{Tr}\left[\varrho^2\right] \le 1$ the coefficients s_{nm} must obey the further condition

$$\sum_{n=0}^{d-1}\sum_{m=0}^{d-1} |s_{nm}|^2 \le d. \tag{21}$$

Notice that the two-dimensional case is recovered by taking $U(0,0) = \sigma_0$, $U(0,1) = \sigma_1$, $U(1,1) = -i\sigma_2$, $U(1,0) = \sigma_3$.

In order to simplify the calculation of probabilities we describe the probe state in the basis $U^*(p,q)$ i.e. $\tau = \frac{1}{d}\sum_{pq} t_{pq} U^*(p,q)$. The signal/probe state is thus given by

$$\varrho \otimes \tau = \frac{1}{d^2}\sum_{nm}\sum_{pq} s_{nm}\, t_{pq}\, U(n,m) \otimes U^*(p,q). \tag{22}$$

The probability of the outcome λ_{kl} in the measurement of the d-dimensional Bell observable (17) is given by

$$\begin{aligned} p_{kl} &= \langle\langle \frac{1}{\sqrt{d}} U(k,l)|\varrho\otimes\tau|\frac{1}{\sqrt{d}}U(k,l)\rangle\rangle \\ &= \frac{1}{d}\langle\langle U(k,l)|\varrho U(k,l)\tau^T\rangle\rangle \\ &= \frac{1}{d^3}\langle\langle U(k,l)|\sum_{nm}\sum_{pq} s_{nm} U(n,m) U(k,l) t_{pq} U^\dagger(p,q)\rangle\rangle \\ &= \frac{1}{d^3}\sum_{nm}\sum_{pq} s_{nm} t_{pq} \\ &\quad \times \text{Tr}\left[U^\dagger(k,l) U(n,m) U(k,l) U^\dagger(p,q)\right]. \end{aligned} \tag{23}$$

Using equations (15), (52) and (16), equation (23) rewrites as

$$\begin{aligned} p_{kl} &= \frac{1}{d^3}\sum_{nm}\sum_{pq} s_{nm} t_{pq} \\ &\quad\times \text{Tr}\left[e^{-\frac{2\pi}{d}l(n-k)} U(n-k,m-l) U(k,l) U^\dagger(p,q)\right] \\ &= \frac{1}{d^3}\sum_{nm}\sum_{pq} s_{nm} t_{pq} \\ &\quad\times \text{Tr}\left[e^{-\frac{2\pi}{d}l(n-k)} e^{\frac{2\pi}{d}k(m-l)} U(n,m) U^\dagger(p,q)\right] \\ &= \frac{1}{d^3}\sum_{nm}\sum_{pq} s_{nm} t_{pq} e^{\frac{2\pi}{d}(km-ln)} \text{Tr}\left[U(n,m)U^\dagger(p,q)\right] \\ &= \frac{1}{d^3}\sum_{nm}\sum_{pq} s_{nm} t_{pq} e^{\frac{2\pi}{d}(km-ln)} \left(d\delta_{np}\delta_{mq}\right), \end{aligned} \tag{24}$$

and finally

$$p_{kl} = \frac{1}{d^2}\sum_{pq} s_{pq}\, t_{pq}\, e^{\frac{2\pi i}{d}(kq-lp)}. \tag{25}$$

Equation (25) is a linear system in the d^2 unknowns $\{s_{nm}\}$, which can be solved upon the assumption $t_{nm} \neq 0$ $\forall n, \forall m$. By inverse discrete Fourier transform we get

$$s_{mn} = \frac{1}{d^2}\frac{1}{t_{mn}}\sum_{kl} p_{kl}\, e^{\frac{2\pi i}{d}(lm-nk)}. \tag{26}$$

Analogously to the bidimensional case, also state reconstruction for qudits may be based on any other Bell measurement, whereas factorized measurements are useless for this purpose.

5 Reconstruction of the expectation value of generic operator

So far we have considered the reconstruction of the elements of the density matrix. If the quantity of interest is the expectation value of a generic operator X we may obtain it according to the Born rule $\langle X\rangle = \text{Tr}\left[\varrho X\right]$, by expanding X in the basis used for the reconstruction of the density matrix. On the hand, it is possible to obtain $\langle X\rangle$ directly from the experimental data, without the in-between step of reconstructing the matrix elements. This result, which will be also useful in discussing statistical errors, may be achieved by introducing a set of functions, called *pattern functions*, which gives the desired expectation value when averaged over the outcomes of the Bell measurement. This procedure has been referred to as universal detection [11,24]. In this Section we focus on the derivation of the pattern function for a generic operator X, which may also correspond to quantities that are not directly observable. Our derivation is similar to that of reference [24]. Pattern functions depend on the probe state τ and on the operator X, and are defined through the relation

$$\mathrm{Tr}\,[\varrho X] \equiv \sum_{\nu=0}^{d^2-1} R_\nu[\tau, X] p_\nu = \overline{R\,[\tau, X]} \qquad (27)$$

where p_ν is again the probability of the outcome λ_ν. Throughout this section $\{U_\nu\}$ denote the set of unitaries involved in the Bell measurement, labeled by a single poly-index in order to simplify notation.

In order to get an expression for the pattern function in terms of known quantities only we will use few other properties of the Hilbert spaces $\mathcal{H}_1 \otimes \mathcal{H}_2$ and $\mathcal{L}[\mathcal{H}_1 \to \mathcal{H}_2]$. At first we expand the generic operator X using the unitaries U_ν as a basis:

$$X = \frac{1}{d} \sum_{\lambda=0}^{d^2-1} \mathrm{Tr}\left[X U_\lambda^\dagger\right] U_\lambda, \qquad (28)$$

which also implies the relation

$$\frac{1}{d} \sum_{\lambda=0}^{d^2-1} U_\lambda\, X\, U_\lambda^\dagger = \mathbb{I}\mathrm{Tr}\,[X]\,. \qquad (29)$$

In order to prove (29) let us introduce the operator $B = \sum_{\lambda=0}^{d^2-1} U_\lambda\, X\, U_\lambda^\dagger$ and notice that $\mathrm{Tr}\,[B] = d^2 \mathrm{Tr}\,[X]$. In addition, using equation (15) we have

$$\begin{aligned} U_\nu B &= \sum_{\lambda=0}^{d^2-1} U_\nu U_\lambda X U_\lambda^\dagger U_\nu^\dagger U_\nu = \sum_{\lambda}^{d^2-1} U_\mu e^{i\alpha_{\nu\lambda}} X U_\mu^\dagger e^{-i\alpha_{\nu\lambda}} U_\nu \\ &= \sum_{\mu=0}^{d^2-1} U_\mu X U_\mu^\dagger U_\nu = B U_\nu \quad \forall \lambda, \end{aligned} \qquad (30)$$

that is, B commutes with the set of unitaries U_ν. As a consequence, using *Schur's lemma*, we have $B = \lambda\mathbb{I}$ and thus $\mathrm{Tr}\,[B] = d\lambda$ which, in turn, leads to $\lambda = d\mathrm{Tr}\,[X]$ and to equation (29).

The composition property of the unitaries U_ν will be also useful. Using equations (15) and (52) one arrives at

$$U_\lambda U_\nu U_\lambda^\dagger = U_\nu e^{-i\phi_{\lambda\nu}} \qquad (31)$$

which also implies

$$e^{-i\phi_{\lambda\nu}} = \frac{1}{d}\,\mathrm{Tr}\left[U_\nu^\dagger U_\lambda U_\nu U_\lambda^\dagger\right]. \qquad (32)$$

We are now in the position of deriving a compact formula for the pattern functions (27). Starting from (28):

$$\begin{aligned} \mathrm{Tr}\,[\varrho X] &= \frac{1}{d} \sum_{\lambda=0}^{d^2-1} \mathrm{Tr}\left[X U_\lambda^\dagger\right] \mathrm{Tr}\,[\varrho U_\lambda] \\ &= \frac{1}{d} \sum_{\lambda=0}^{d^2-1} \frac{\mathrm{Tr}\left[X U_\lambda^\dagger\right]}{\mathrm{Tr}\left[U_\lambda^\dagger \tau^T\right]}\, \mathrm{Tr}\left[\varrho U_\lambda\, \mathrm{Tr}\left[U_\lambda^\dagger \tau^T\right]\right] \\ &= \frac{1}{d^2} \sum_{\lambda=0}^{d^2-1} \frac{\mathrm{Tr}\left[X U_\lambda^\dagger\right]}{\mathrm{Tr}\left[U_\lambda^\dagger \tau^T\right]}\, \mathrm{Tr}\left[\varrho U_\lambda \sum_{\nu=0}^{d^2-1} U_\nu U_\lambda^\dagger \tau^T U_\nu^\dagger\right] \\ &= \frac{1}{d^2} \sum_{\lambda\nu} \frac{\mathrm{Tr}\left[X U_\lambda^\dagger\right]}{\mathrm{Tr}\left[U_\lambda^\dagger \tau^T\right]}\, \mathrm{Tr}\left[\varrho U_\lambda U_\nu U_\lambda^\dagger \tau^T U_\nu^\dagger\right] \\ &= \frac{1}{d^2} \sum_{\lambda\nu} \frac{\mathrm{Tr}\left[X U_\lambda^\dagger\right]}{\mathrm{Tr}\left[U_\lambda^\dagger \tau^T\right]}\, e^{-i\phi_{\lambda\nu}}\, \mathrm{Tr}\left[\varrho U_\nu \tau^T U_\nu^\dagger\right]. \end{aligned} \qquad (33)$$

Finally, using

$$\begin{aligned} \mathrm{Tr}\left[\varrho U_\nu \tau^T U_\nu^\dagger\right] &= \mathrm{Tr}\left[|\varrho U_\nu \tau^T\rangle\rangle\langle\langle U_\nu|\right] \\ &= \mathrm{Tr}\left[\varrho \otimes \tau |U_\nu\rangle\rangle\langle\langle U_\nu|\right] = d\, p_\nu \end{aligned} \qquad (34)$$

we arrive at

$$R_\nu[\tau, X] = \frac{1}{d} \sum_{\lambda=0}^{d^2-1} \frac{\mathrm{Tr}\left[X\, U_\lambda^\dagger\right]}{\mathrm{Tr}\left[U_\lambda^\dagger\, \tau^T\right]}\, e^{-i\phi_{\lambda\nu}}. \qquad (35)$$

For any given operator X the pattern function $R_\nu[\tau, X]$ can be evaluated through equation (35), upon the knowledge of the state of the probe. The latter is subjected to the constrain $\mathrm{Tr}[U_\lambda^\dagger \tau^T] \neq 0$ for any λ such that $\mathrm{Tr}[X U_\lambda] \neq 0$, i.e the probe should have a non zero component in any subspace where the operator of interest is non zero.

6 Optimization of the pattern functions

Using equation (27) one has that the expectation value of the operator X may be obtained by evaluating the corresponding pattern function through equation (35) and then averaging it over the outcomes of the Bell measurement. In other words, the function $R[\tau, X]$ is the statistical *estimator* for the quantity $\langle X\rangle$. Suppose that N experimental runs have been performed, then the estimation of the expectation value is given by the sample average

$$\overline{X} \equiv \overline{R[\tau, X]} = \sum_\nu p_\nu R_{\nu_j}[\tau, X] \longrightarrow \sum_{j\in data} R_{\nu_j}[\tau, X], \qquad (36)$$

whereas the confidence interval on this determination corresponds (since the estimators satisfy the hypothesis of the central limit theorem) to the rms deviation of the estimator divided by the square root of the number of runs, i.e.

$$\delta X = \frac{1}{\sqrt{N}} \sqrt{\overline{\Delta R^2[\tau, X]}} \qquad (37)$$

where $\overline{\Delta R^2[\tau, X]} = \overline{R^2[\tau, X]} - \overline{\Delta R[\tau, X]}^2$ and $\overline{R^2[\tau, X]} = \sum_\nu p_\nu R^2_{\nu_j}[\tau, X] \rightarrow \sum_{j\in data} R^2_{\nu_j}[\tau, X]$. A question arises on whether it is possible to minimize the confidence interval for any given operator by a suitable preparation of the probe. Notice that being the unitaries U_ν a basis for the space of operators the pattern function for a given operator is unique and there are no null functions [31]. In order to optimize pattern functions let us start by expanding the operator X in the U_ν basis as $X = \sum_\nu x_\nu U_\nu$ and take the derivative of the rms deviation with respect to the probe matrix elements

$$\frac{\partial}{\partial t_p}\overline{\Delta R^2[\tau, X]} = \sum_\nu \left[2p_\nu R_\nu[\tau, X]\frac{\partial R_\nu[\tau, X]}{\partial t_p} + R^2_\nu[\tau, X]\frac{\partial p_\nu}{\partial t_p} + 2\overline{R[\tau, X]}\left(p_\nu \frac{\partial R_\nu[\tau, X]}{\partial t_p} + R_\nu[\tau, X]\frac{\partial p_\nu}{\partial t_p}\right)\right] \quad (38)$$

$$= \frac{1}{4}\left[s_p \left(1 + 2\overline{R[\tau, X]}\right) A - 4\frac{x_p}{t_p^2}\left(B + \overline{R[\tau, X]}C\right)\right] \quad (39)$$

where

$$A = \sum_\nu \exp\left[-i\phi_{p\nu}\right] R_\nu[\tau, X] \quad (40)$$

$$B = \sum_\nu \exp\left[-i\phi_{p\nu}\right] p_\nu\, R_\nu[\tau, X] \quad (41)$$

$$C = \sum_\nu \exp\left[-i\phi_{p\nu}\right] p_\nu \quad (42)$$

with $\exp\left[-i\phi_{p\nu}\right]$ given in equation (32). By solving the systems of nonlinear equations

$$\frac{\partial}{\partial t_p}\overline{\Delta R^2[\tau, X]} = 0 \qquad p = 1, 2, 3, \quad (43)$$

one arrives at the optimal probe preparation to minimize statistical errors in the reconstruction.

As an example, let us know reconsider the reconstruction of the density matrix for a qubit system. The pattern functions for the three components of the signal Bloch vector, i.e. for the expectation values of the three Pauli matrices σ_k, and the corresponding rms deviations, are given by

$$R_\nu[\tau, \sigma_k] = \frac{(-1)^{\delta_{k\nu}+\delta_{\nu 2}}}{t_k} \quad (44)$$

$$\overline{\Delta R^2[\tau, \sigma_k]} = -s_k^2 + \frac{1}{t_k^2}. \quad (45)$$

As it is apparent from equation (45) an optimized choice τ^* for the probe preparation in order to minimize the error in the reconstruction of $\langle\sigma_k\rangle \equiv s_k$ is given by $t_k = \pm 1$ and $t_j = 0$ for $j \neq k$, independently on the signal under investigation. Notice that in this case the rms deviation $\overline{\Delta R^2[\tau^*, \sigma_k]}$ equals the intrinsic quantum uncertainty $\overline{\Delta\sigma_k^2} = \langle\sigma_k^2\rangle - \langle\sigma_k\rangle^2 = 1 - s_k^2$ of a spin measurement in a given direction, i.e. the noise added by the indirect reconstruction with respect to the direct measurement of the same quantity [32] vanishes. If one is interested in reconstructing the whole density matrix, the prescriptions to optimize the pattern functions for the three Bloch components are mutually incompatible and one is faced with the problem of choosing the best strategy. A first strategy could be that of a "balanced" probe preparation $t_k^2 = 1/3$, which is suboptimal for each component, but nevertheless allows to use each datum for the reconstruction of all the components. The confidence intervals on the reconstructions are given by

$$\delta\sigma_k = \sqrt{\frac{3 - s_k^2}{N}}. \quad (46)$$

On the other hand one may allocate one third of the runs to the optimal reconstruction of each components (and, in turn, without using the same data to reconstruct the other components). The corresponding confidence intervals are then given by

$$\delta\sigma_k = \sqrt{\frac{3(1 - s_k^2)}{N}}. \quad (47)$$

As it is apparent from equations (46) and (47) the second choice is always convenient, independently on the signal under investigation i.e. the optimization is useful to minimize statistical errors.

Concerning quantities that are not observables we consider, as an example the reconstruction of the expectation value of the operator $\sigma_+ = (\sigma_1 + i\sigma_2)/2$ on a qubit system. We have

$$R_\nu[\tau, \sigma_+] = (-1)^{\delta_{\nu 0}}\left[i\,\frac{(-1)^{\delta_{\nu 2}}}{t_2} - \frac{(-1)^{\delta_{\nu 1}}}{t_1}\right]\cdot \quad (48)$$

Since $\langle\sigma_+\rangle$ is a complex quantity, confidence intervals should be provided on the real and the imaginary part separately, which in turn correspond to components of the Bloch vectors already discussed above. Finally, for the generic combination $\sigma_\alpha = \sin\alpha\sigma_1 + \cos\alpha\sigma_2$ we have

$$\overline{\Delta R^2[\tau, \sigma_\alpha]} = -(s_1 \sin\alpha + s_2\cos\alpha)^2 + \frac{\cos^2\alpha}{t_2^2} + \frac{\sin^2\alpha}{t_1^2} - 2\sin\alpha\cos\alpha\frac{t_3}{t_1 t_2}. \quad (49)$$

Since the present reconstruction method is devised to work without any a priori information we average the rms deviation in equation (49) over the whole Bloch sphere and look for an optimized probe preparation valid for an unknown input signal. In this case one easily prove that the optimal probe τ^* is defined by $t_3 = 0$, $t_1^2 = t_2^2 = 1/2$, for which we have $\overline{\Delta R^2[\tau^*, \sigma_\alpha]} = 4 - (s_1\sin\alpha + s_2\cos\alpha)^2$.

Notice that the optimization above are suitable for the measurement of the specific observable under investigation. In cases when several observables are of interest at the same time the optimal probe should be a balanced superposition. If the set of observables is large, then the optimal probe approaches the pure balanced superposition, $t_1 = t_2 = t_3 = 1/\sqrt{3}$, representing the state *most spread* in the Hilbert space.

In order to check the statistical reliability of our method we have performed several Monte Carlo simulated experiments. Here we report few results for qubit systems. The aim is twofold: on the one hand we prove the statistical reliability of the method by checking the scaling of the confidence interval versus the number of runs and probe parameters and, on the other hand, we show the effectiveness of the optimization method developed above. We focused on the operator $\sigma_x = \sigma_1$. We fixed the signal state in the eigenstate $|+\rangle_x$ of σ_x, corresponding to the Bloch vector $(1, 0, 0)$. For the probe we used three different states: the *almost ideal pure state* $t_1 = 0.9999$, $t_2 = t_3 = \frac{1}{\sqrt{2}}\sqrt{1 - 0.9999^2}$ (in order not to violate the condition $t_i \neq 0$), the *pure balanced state* $t_1 = t_2 = t_3 = 1/\sqrt{3}$ and a generic *mixed state* defined by the Bloch vector (0.1, 0.3, 0.2). The number of runs in the simulated experiments ranges from 10^4 to 10^5 and for each set of runs we calculated the mean value, that is the expectation value for σ_x on the state $|+\rangle_x$, and the confidence interval δ_x.

The results are reported in Figure 2. As it is apparent from the plot there is an excellent agreement, either as concern the confidence interval, as well as for the scaling versus the number of runs. We then simulated 10^5 measurements with the same signal and the almost ideal probe state and we repeated the process 10 times (changing the random seed) arriving at $\delta_x = (5 \pm 1) \times 10^{-5}$ in agreement with the theoretical value $\delta_x = 4 \times 10^{-5}$, as calculated from equation (45).

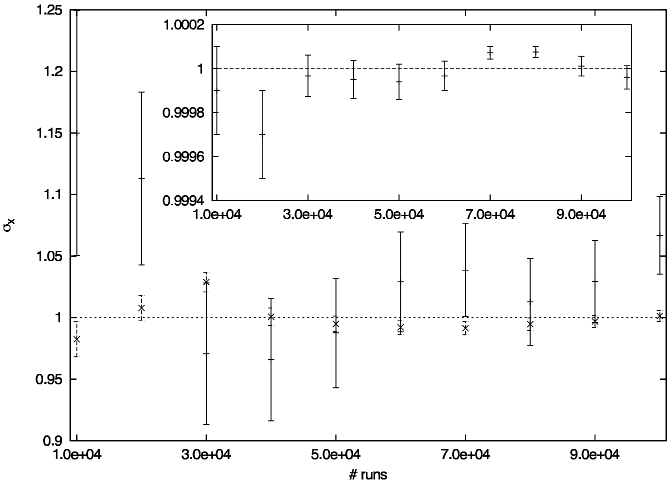

Fig. 2. Monte Carlo simulated experiments for the reconstruction of the expectation value of the operator σ_x on the qubit signal state $|+\rangle_x$. The dashed line with $\times$ symbols corresponds to the reconstruction obtained using the *pure balanced state* $t_1 = t_2 = t_3 = 1/\sqrt{3}$ as a probe qubit, whereas the continuous line with + symbols corresponds to the probe prepared in *mixed state* (0.1, 0.3, 0.2). In the inset we show the results for the probe in the *almost ideal pure state* $t_1 = 0.9999$, $t_2 = t_3 = \sqrt{1 - 0.9999^2}/\sqrt{2}$. In both plots the horizontal line represents the theoretical expectation value ($\langle\sigma_x\rangle = 1$).

7 Conclusions

We have addresses quantum state reconstruction for d-dimensional systems assisted by entangled measurements. In our scheme the system of interest is jointly detected together with a probe through the measurement of a single entangled observable defined on the bipartite system/probe Hilbert space. We have shown that the statistics of the measurement and the knowledge of the probe preparation suffice to reliably reconstruct the density matrix of the system, as well as the expectation value of any desired operator. Being suitable to reconstruct the expectation value of *all* the system observables, our method is subjected to larger statistical errors with respect to the direct measurement of each observables. On the other hand, the statistical errors in the reconstruction may be optimized by tuning the probe preparation, thus minimizing the added noise.

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Appendix: Properties of the unitaries U(n, m)

As concerns the operator trace, we have

$$\mathrm{Tr}\left[U(n,m)\right] = \sum_{h=0}^{d-1}\sum_{k=0}^{d-1} e^{2\pi i k n/d}\langle h|k\rangle\langle k\oplus m|h\rangle = d\delta_{n0}\,\delta_{m0}.$$

The condition for non-zero trace is therefore $n, m = 0$ and of course $U(0, 0) = \mathbb{I}$, $\mathrm{Tr}\left[U(0, 0)\right] = d$. Consider now the composition $U(n, m)U(p, q)$. We have

$$\begin{aligned}
U(n,m)U(p,q) &= \sum_{k=0}^{d-1} e^{2\pi i k n/d}|k\rangle\langle k\oplus m| \\
&\quad\times \sum_{h=0}^{d-1} e^{2\pi i h p/d}|h\rangle\langle h\oplus q| \\
&= \sum_{k=0}^{d-1}\sum_{h=0}^{d-1} e^{2\pi i k n/d} e^{2\pi i h p/d}|k\rangle\langle h\oplus q|\delta_{k+m,h} \\
&= \sum_{k=0}^{d-1} e^{2\pi i k(n+p)/d} e^{2\pi i m p/d}|k\rangle\langle k\oplus(m+q)| \\
&= e^{\frac{2\pi i}{d}mp}\, U(n\oplus p, m\oplus q)
\end{aligned} \tag{50}$$

thus leading equation (15). Concerning orthogonality, we have from equation (3) that $\langle\langle U(p, q)|U(n, m)\rangle\rangle = \mathrm{Tr}\left[U^\dagger(p, q)U(n, m)\right]$. Therefore, one can write $U^\dagger(p, q)$ as

$$\begin{aligned}
U^\dagger(p,q) &= \sum_{h=0}^{d-1} e^{-\frac{2\pi i}{d}hp}|h+q\rangle\langle h| \\
&= \sum_{n=0}^{d-1} e^{-\frac{2\pi i}{d}(n-q)p}|n\rangle\langle n-q| \\
&= \sum_{n=0}^{d-1} e^{-\frac{2\pi i}{d}np} e^{\frac{2\pi i}{d}pq}|n\rangle\langle n-q| \\
&= e^{\frac{2\pi i}{d}pq}U(-p,-q).
\end{aligned} \tag{51}$$

This result, together with the composition rule (15) leads to

$$U^\dagger(p,q)U(n,m) = e^{\frac{2\pi i}{d}pq}\, e^{-\frac{2\pi i}{d}qn}\, U(n-p,m-q)$$
$$= e^{\frac{2\pi i}{d}\,q\,(p-n)}\, U(n-p,m-q) \quad (52)$$

where the subtraction of the indexes still obeys to the group equivalence relations. Since only $U(0,0)$ has non-zero trace one finally arrives at equation (16).

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